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HYPONORMAL TOEPLITZ OPERATORS AND EXTREMAL PROBLEMS OF HARDY SPACES

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Dedicated to Professor Tsuyoshi Ando on his sixtieth birthday

ABSTRACT. The symbols of hyponormal Toeplitz operators are completely described and those are also studied, being related with the extremal problems of Hardy spaces. Moreover, we discuss Halmos's question about a subnormal Toeplitz operator when the self-commutator is finite rank.

1. INTRODUCTION

Let L^p be the Lebesgue space on the unit circle Γ and let H^p be the corresponding Hardy space for $1 \le p \le \infty$. The Toeplitz operator T_ϕ with symbol ϕ in L^∞ is the operator on H^2 defined by $T_\phi x = P(\phi x)$ for x in H^2, where P is the orthogonal projection of L^2 onto H^2. Brown and Halmos [3] began the systematic study of the algebraic properties of Toeplitz operators and showed that T_ϕ is normal if and only if $\phi = \alpha u + \beta$ where α and β are complex numbers and u is a real-valued function in L^∞. A characterization of symbols of hyponormal Toeplitz operators is known [5]. An operator A is called hyponormal if its self-commutator $[A^*, A] = A^*A - AA^*$ is positive. However the exact descriptions of symbols of hyponormal Toeplitz operators are not known. The main purpose of this paper is to give such descriptions as those of Brown and Halmos for normal Toeplitz operators. The symbol of a hyponormal Toeplitz operator T_ϕ satisfies that $\phi - g = k\overline{\phi}$, $g \in H^\infty$, and $k \in H^\infty$ with $\|k\|_\infty \le 1$ (see Proposition 1). Therefore $|\phi - g| \le |\phi|$ a.e. on Γ, and hence our study is related with the extremal problems of Hardy spaces H^∞ and H^1. In his paper "Ten problems in Hilbert space" [9], Halmos raised the question: "Is every subnormal Toeplitz operator either normal or analytic?" A Toeplitz operator T_ϕ is analytic if its symbol ϕ is in H^∞. Cowen and Long [6] answered negatively Halmos's question. Abrahamse [1] gave a very general sufficient condition for the answer to be yes, after previous works of [10] and [2]. In this paper we give a simple proof of Abrahamse's theorem and a new sufficient condition for the answer to be yes. A subnormal operator is always

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hyponormal. Hence it is desirable to describe symbols of subnormal Toeplitz operators in general.

The paper is arranged as follows. In §2, we give the general descriptions of symbols of hyponormal Toeplitz operators. In §§3 and 4, we study the relation between the study of the symbols of hyponormal Toeplitz operators and the extremal problems of Hardy spaces. In §5, we study Toeplitz operators of symbols of bounded type. In §§6 and 7, the self-commutators of hyponormal Toeplitz operators are studied. In §8, we discuss a positive answer for Halmos's question.

Throughout this paper we use the following lemma which was essentially proved by Cowen [5].

Lemma 1. *T_ϕ is hyponormal if and only if there exist two functions k and g in H^∞ such that $\phi = k\overline{\phi} + g$ and $\|k\|_\infty \le 1$.*

Proof. Let $\phi \in L^\infty$ and $\phi = h + \overline{l}$ where h and l are in H^2. Cowen [5, Theorem 1] showed that T_ϕ is hyponormal if and only if $l = c + T_{\overline{k}}h$ for some constant c and some function k in H^∞ with $\|k\|_\infty \le 1$. If $\phi = k\overline{\phi} + g$, $g \in H^\infty$, and $k \in H^\infty$ with $\|k\|_\infty \le 1$, then

$$\phi - k\overline{\phi} = \overline{l} - k\overline{h} + h - kl.$$

Therefore $\overline{l} - k\overline{h} \in H^2$ and hence $l = c + T_{\overline{k}}h$ for some constant c. Thus T_ϕ is hyponormal. For the converse the proof above is reversible.

2. General description of symbols

In this section, the general descriptions of symbols of hyponormal Toeplitz operators are given. Two kinds of symbols of Toeplitz operators in Propositions 1 and 2 are typical and the arbitrary symbol is a combination of these two symbols. If q is an inner function then $q = (1+q)^2/|1+q|^2$. We define $q^{1/2}$ as in the following:

$$q^{1/2} = (1+q)/|1+q|.$$

For an arbitrary nonzero function k in H^∞, if q is the inner part and h is the outer part then put $k^{1/2} = q^{1/2}h^{1/2}$. We should note that $(z^2)^{1/2} \ne z$. For a measurable set E of Γ, χ_E denotes the characteristic function of E.

Proposition 1. *If $\phi = q^{1/2}u + g$ where q is inner, u is a real function in L^∞, and g is a function in H^∞ with $q\overline{g}$ in H^∞, then T_ϕ is hyponormal.*

Proof. Since $\phi - q\overline{\phi} = g - q\overline{g}$ and $q\overline{g} \in H^\infty$, $\phi - q\overline{\phi}$ belongs to H^∞. Lemma 1 implies that T_ϕ is hyponormal.

Proposition 2. *If $\phi = (k\overline{g} + g)/(1 - |k|^2)$ is bounded where k and g are functions in H^∞ and $|k| < 1$ a.e., T_ϕ is hyponormal.*

Proof. A calculation implies that $\phi - k\overline{\phi} = g$ and hence by Lemma 1 T_ϕ is hyponormal.

Theorem 3. *Let ϕ be a nonzero function in L^∞. T_ϕ is hyponormal if and only if*

$$\phi = \chi_E(k^{1/2}u + \tfrac{1}{2}g) + \chi_{E^c}\frac{k\overline{g} + g}{1 - |k|^2},$$

where k and g are functions in H^∞ with $\|k\|_\infty \le 1$, $E = \{e^{i\theta} \in \Gamma : |k(e^{i\theta})| = 1\}$, $k\overline{g} = -g$ a.e. on E and u is a real-valued function in L^∞.

Proof. For the 'if' part, by a simple calculation

$$\phi - k\overline{\phi} = \chi_E(\tfrac{1}{2}g - \tfrac{1}{2}k\overline{g}) + \chi_{E^c}g = g$$

because $k\overline{g} = -g$ a.e. on E.

We will show the 'only if' part. Lemma 1 shows that $\phi = k\overline{\phi} + g$ where $g \in H^\infty$ and $k \in H^\infty$ with $\|k\|_\infty \le 1$. Put $E = \{e^{i\theta} \in \Gamma : |k(e^{i\theta})| = 1\}$. If $g = 0$ a.e., then $\phi^2 = k|\phi|^2$ and hence $\phi = \chi_E k^{1/2}u$ where u is a real-valued function in L^∞. We will assume that g is nonzero. Since $\phi = k(\overline{k}\phi + \overline{g}) + g$, $(1 - |k|^2)\phi = k\overline{g} + g$. This implies that on E^c

$$\phi = (k\overline{g} + g)/(1 - |k|^2)$$

and on E, $k\overline{g} + g = 0$ and hence $ig = k^{1/2}|g|(2\chi_{E_0} - 1)$, where E_0 is a measurable set of Γ. Since $\phi g^{-1} - k\overline{\phi}g^{-1} = 1$,

$$\phi/g + \overline{\phi}/\overline{g} = 1 \quad \text{on } E$$

because $k\overline{g} = -g$ on E. Then

$$\phi = (ig)\mathrm{Im}(\phi g^{-1}) + \tfrac{1}{2}g = k^{1/2}u + \tfrac{1}{2}g$$

and $u = |g|\,\mathrm{Im}(\phi g^{-1})(2\chi_{E_0} - 1)$ because $ig = k^{1/2}|g|(2\chi_{E_0} - 1)$. This completes the proof.

In Theorem 3 ϕ has the different forms on E and E^c. We can expect that $d\theta(E) = 2\pi$ or $d\theta(E^c) = 2\pi$. However this is not the case by the following example. If $k = \exp(\chi_E - 1 + iv)$ then $\|k\|_\infty \le 1$, $|k| = 1$ a.e. on E and $|k| = e^{-1}$ a.e. on E^c, where v is a harmonic conjugate of χ_E and $0 < d\theta(E) < 2\pi$. Put $g = ik^{1/2}$. Then

$$\phi = \chi_E(k^{1/2} + \tfrac{1}{2}g) + \chi_{E^c}(k\overline{g} + g)(1 - |k|^2)^{-1}$$

is bounded and $k\overline{g} + g = 0$ on E. Moreover,

$$\phi = k^{1/2}\{(1 + \tfrac{1}{2}i)\chi_E + i(1 - e^{-1})(1 - e^{-2})^{-1}\chi_{E^c}\}.$$

Therefore ϕ is not of bounded type and hence this representation is unique (see §6). In Proposition 2, if $\|k\|_\infty < 1$ then ϕ is bounded for arbitrary g in H^∞. In Proposition 1, if $q = z$ then g is a polynomial of degree at most one. For arbitrary k in H^∞, we want to know such functions g. For $k \in H^\infty$ with $\|k\|_\infty \le 1$, put

$$\mathscr{H}(k) = \{g \in H^\infty : \phi = k\overline{\phi} + g \text{ for some } \phi \in L^\infty\}.$$

By [5, Theorem 1] we can describe $\mathscr{H}(k)$ as

$$\mathscr{H}(k) = \{g \in H^\infty : g = f - k\overline{f} + \overline{T_{\overline{k}}f} - k(T_{\overline{k}}f) \text{ for some } f \in H^2\}.$$

The following corollary gives another description of $\mathscr{H}(k)$.

Corollary 1. *Let k be in H^∞ and $\|k\|_\infty \le 1$. Then $\mathscr{H}(k)$ is a nonzero real subspace of H^∞ and*

$$\mathscr{H}(k) = \{\alpha g : |g + k\overline{g}| + |k| \le 1 \text{ a.e.}, g \in H^\infty \text{ and } \alpha \in \mathbb{R}\}.$$

If $\|k\|_\infty < 1$ *then* $\mathscr{H}(k) = H^\infty$ *and if* k *is inner then*

$$\begin{aligned}\mathscr{H}(k) &= \{g \in H^\infty : k\overline{g} = -g\} \\ &= \{g - k\overline{g} : g \in H^2 \ominus kzH^2 \text{ and } g \in H^\infty\}.\end{aligned}$$

Proof. It is clear that $\mathscr{H}(k)$ is a real subspace and $1 - k \in \mathscr{H}(k)$. Since $(1 - |k|^2)\phi = k\overline{g} + g$ by the proof of Theorem 3, $\mathscr{H}(k)$ has the form above. Hence if $\|k\|_\infty < 1$ then $\mathscr{H}(k) = H^\infty$ trivially. If k is inner then $|g + k\overline{g}| \leq 1 - |k| = 0$ and hence $g = -k\overline{g}$. Thus $\mathscr{H}(k) = \{g \in H^\infty : k\overline{g} = -g\}$. If $g_0 = g - k\overline{g}$ where $g \in H^2 \ominus kzH^2$ and $g \in H^\infty$, then $k\overline{g}_0 = -g_0$ and $g_0 \in \mathscr{H}(k)$. If $g_0 \in \mathscr{H}(k)$ then $k\overline{g}_0 = -g_0$ and hence $g_0 \in H^2 \ominus kzH^2$. Put $g = g_0/2$; then $g_0 = g - k\overline{g}$ and $g \in H^2 \ominus kzH^2$. Therefore $\mathscr{H}(k) = \{g - k\overline{g} : g \in H^2 \ominus kzH^2 \text{ and } g \in H^\infty\}$.

In the following corollary the equivalence between (1) and (2) is clear by Theorem 3. Statement (3) shows that such a hyponormal Toeplitz operator is the product of a normal Toeplitz (possibly unbounded) and an analytic Toeplitz operator (hence subnormal).

Corollary 2. *Let* ϕ *be a function in* L^∞. *Then the following are equivalent.*

(1) $\phi = q\overline{\phi} + g$ *where* q *is inner and* $g \in H^\infty$.

(2) $\phi = q^{1/2}u + g$ *where* u *is a real-valued function in* L^∞, q *is inner, and* $g \in H^\infty$ *with* $q\overline{g} = -g$.

(3) $\phi = (v + c)f$ *where* v *is a real-valued measurable function,* c *is a complex number, and* $f \in H^\infty$ *with* $q\overline{f} = f$.

Proof. We will show only the equivalence between (2) and (3).

(2) $\Rightarrow$ (3). For some measurable subset E of Γ

$$q^{1/2} = ig(2\chi_E - 1)/|g|$$

because $q\overline{g} = -g$. Put $v = u(2\chi_E - 1)/|g|$, $f = ig$, and $c = -i$; then $\phi = (v + c)f$.

(3) $\Rightarrow$ (2). Let $c = \alpha + i\beta$ where α and β are real. For some measurable subset E of Γ, $f/|f| = q^{1/2}(2\chi_E - 1)$ because $q\overline{f} = f$. Put $u = |f|(v + \alpha)(2\chi_E - 1)$ and $g = i\beta f$; then $\phi = q^{1/2}u + g$.

3. Extremal problems in Hardy spaces

The equation of a hyponormal symbol ϕ: $\phi = k\overline{\phi} + g$ is related to the extremal problem in H^∞ (and hence H^1) (cf. [8, Chapter IV]) because $|\phi - g| \leq |\phi|$ a.e. on Γ. When $\phi = k\overline{\phi} + g$ and $\phi \neq 0$ a.e., the following are equivalent: (1) $g = 0$ a.e., (2) $\phi = k\overline{\phi}$, (3) $\phi = k^{1/2}u$ where k is inner and u is a real-valued function in L^∞, and (4) $\phi = fv$ where v is a real-valued measurable function and $f \in H^\infty$ with $k\overline{f} = f$. When does the symbol ϕ satisfy one of the equivalent conditions? Such Toeplitz operators have been studied in [4]. We answer this question in the following theorem.

Theorem 4. *Let* ϕ *be in* L^∞ *and* $\phi \neq 0$ *a.e.*

(1) *When* $\log|\phi|$ *is not integrable,* T_ϕ *is hyponormal if and only if* $\phi = q\overline{\phi}$ *for some inner* q.

(2) *Suppose* T_ϕ *is hyponormal. If there does not exist a nonzero function* g *in* H^∞ *with* $|\phi - g| \leq |\phi|$ *a.e. then* $\phi = q\overline{\phi}$ *for some inner* q.

(3) *Suppose* T_ϕ *is hyponormal and nonnormal. If* $\phi = q\overline{\phi}$ *for some inner* q *then there does not exist a nonzero function* g *in* H^∞ *with* $|\phi - \overline{g}| \leq |\phi|$ *a.e.*

Proof. (1) Suppose T_ϕ is hyponormal and $\log|\phi|$ is not integrable. If ϕ does not have the form $\phi = q\overline{\phi}$ for some inner q, then by the remark above Theorem 4 there exists a nonzero function g in H^∞ such that $|\phi - g| \leq |\phi|$ a.e. Hence $|g|^2 \leq 2|\phi|\,|g|$ and $\log|\phi|$ is integrable. This contradiction shows that $\phi = q\overline{\phi}$ for some inner q. The converse is just Lemma 1.

(2) By the proof of (1) if ϕ does not have the form $\phi = q\overline{\phi}$ for some inner q, then there exists a nonzero function g in H^∞ such that $|\phi - g| \leq |\phi|$ a.e.

(3) If there exists a nonzero function g in H^∞ with $|\phi - \overline{g}| \leq |\phi|$ a.e., then by the proof of (1) $\log|\phi|$ is integrable and hence there exists an outer function k in H^∞ with $|\phi| = |k|$. Put $Q = \overline{\phi}/k$ then $|Q + g'| \leq 1$ a.e. where $g' = g/k$. By a lemma of Koosis (cf. [8, pp. 161–163]) there exists an outer function f in H^1 such that $Q = f/|f|$. If $\phi = q^{1/2}u$ where q is inner and u is a real-valued function in L^∞ then

$$q^{1/2}u/|u| = \phi/|\phi| = |fk|/fk,$$

and hence $q(fk)^2 = |fk|^2$. Therefore $q(fk)^2$ is a nonnegative function in $H^{1/2}$ and hence by [13] $q(fk)^2$ is constant. Since f and k are outer, q is constant and T_ϕ is normal. This contradiction implies (3).

Corollary 3. *Suppose* T_ϕ *is hyponormal and* $\phi \neq 0$ *a.e.*

(1) *If there does not exist an outer function* h *in* H^∞ *such that* $T_\phi^* T_\phi \geq T_h^* T_h$ *then* $\phi = q\overline{\phi}$ *for some inner* q.

(2) *Suppose* T_ϕ *is not normal. If there exists an outer function* h *in* H^∞ *such that* $T_\phi T_\phi^* \geq T_h^* T_h$ *then* ϕ *does not have the form* $\phi = q\overline{\phi}$ *for some nonconstant inner* q.

Proof. By [12, Theorem 2], there exists an outer function h in H^∞ such that $T_\phi^* T_\phi \geq T_h^* T_h$ (or $T_\phi T_\phi^* \geq T_h^* T_h$) if and only if there exists a nonzero function g in H^∞ such that $|\phi - g| \leq |\phi|$ a.e. (or $|\phi - \overline{g}| \leq |\phi|$ a.e.). Now Theorem 4 implies the corollary.

The converses of (2) of Theorem 4 and (1) of Corollary 3 are not true. For example, if ϕ is a positive invertible function in L^∞ then $\phi = \overline{\phi}$ and $T_\phi^* T_\phi = T_\phi T_\phi^* \geq T_c^* T_c$ for some positive constant c and there exists a nonzero function g in H^∞ with $|\phi - g| \leq |\phi|$ a.e. Statement (3) of Theorem 4 (or (2) of Corollary 3) is the weak converse of (2) (or (1), respectively) for nonnormal and hyponormal Toeplitz operators. Now we concentrate on unimodular symbols of Toeplitz operators. In the following theorem (3) is known in the different proof [1, Proposition 2].

Theorem 5. *Let* ϕ *be a unimodular function in* L^∞.

(1) *If* T_ϕ *is hyponormal then* $\phi = f/|f|$ *for some nonzero function* f *in* H^1 *or* $\phi^2 = q$ *for some inner function* q.

(2) *If* $\phi = |f|/f$ *for some nonzero function* f *in* H^1 *whose inverse does not belong to* H^1, *then* T_ϕ *is not hyponormal.*

(3) *If* T_ϕ *is hyponormal and* $\phi = \overline{q}b$ *where* q *and* b *are inner functions, then* $\phi = b_1$ *where* b_1 *is an inner divisor of* b.

(4) *Suppose* $\phi = f/|f|$ *for some nonzero function* f *in* H^1 *and* $\overline{f}$ *is of bounded type. If* T_ϕ *is hyponormal then* $\phi^2 = q$ *for some inner* q.

Proof. (1) If T_ϕ is hyponormal then by Lemma 1 $|\phi - g| \leq 1$ a.e. and $\phi(\phi - g) \in H^\infty$ for some g in H^∞ because $|\phi| = 1$ a.e. If g is nonzero then by a theorem of Koosis (cf. [8, pp. 161–163]) $\phi = f/|f|$ for some nonzero f in H^1. If g is zero then $\phi^2 = q$ for some inner q.

(2) If T_ϕ is hyponormal then by (1) $|f|/f = h/|h|$ for some nonzero h in H^1 or $|f|^2/f^2 = q$ for some inner q. When $|f|/f = h/|h|$, hf is a nonnegative function in $H^{1/2}$ and when $|f|^2/f^2 = q$, qf^2 is a nonnegative function in $H^{1/2}$. Hence by [13] hf is constant or qf^2 is constant. Both contradicts that $f^{-1} \notin H^1$.

(3) Put $\overline{q}b = \overline{q}_1 b_1$ where q_1 and b_1 are relatively prime inner functions. If T_ϕ is hyponormal then by Lemma 1 $\overline{q}_1 b_1 = kq_1\overline{b}_1 + g$ where k and g are in H^∞. Therefore

$$b_1^2 = kq_1^2 + q_1 b_1 g = q_1(kq_1 + b_1 g)$$

and hence q_1 is constant. This implies (3).

(4) If T_ϕ is hyponormal then by Lemma 1

$$f = k\overline{f} + g|f|$$

where k and g are in H^∞. If g is zero then ϕ^2 is an inner function. Suppose g is nonzero. If q is an inner function with $q\overline{f} \in H^1$ then by the equality above $qg|f|$ belongs to H^1. Put $f = bh$ and $g = sl$, where b and s are inner and h and l are outer. Then $qsb|f|/f \in H^\infty$ and hence $\phi = \overline{Q}qsb$ for some inner Q. Now (3) implies (4).

Corollary 4. *Put* $\phi = f/|f|$ *and*

$$f(z) = \prod_{j=1}^{l}(z - \alpha_j)\prod_{j=1}^{m}(z - \beta_j)\prod_{j=1}^{n}(z - \gamma_j)$$

where $|\alpha_j| < 1$ *for* $1 \leq j \leq l$, $|\beta_j| = 1$ *for* $1 \leq j \leq m$, *and* $|\gamma_j| > 1$ *for* $1 \leq j < n$. *Then* T_ϕ *is hyponormal if and only if* $l \geq n$ *and* $\alpha'_j\overline{\gamma}_j = 1$ *for* $1 \leq j \leq n$ *where* $\{\alpha'_j\}_{j=1}^n$ *is a subset of* $\{\alpha_j\}_{j=1}^l$.

Proof. $\phi^2 = f^2/|f|^2 = f/\overline{f}$ and

$$\frac{f}{\overline{f}} = \prod_{j=1}^{m}(-\beta_j)\prod_{j=1}^{n}\frac{\gamma_j}{\overline{\gamma}_j} z^{l+m+n}\prod_{j=1}^{l}\frac{z - \alpha_j}{1 - \overline{\alpha}_j z}\prod_{j=1}^{n}\frac{1 - \gamma_j^{-1}z}{z - \overline{\gamma}_j^{-1}}.$$

By (4) of Theorem 5, T_ϕ is hyponormal if and only if ϕ^2 is inner if and only if

$$\prod_{j=1}^{l}\frac{z - \alpha_j}{1 - \overline{\alpha}_j z}\prod_{j=1}^{n}\frac{1 - \gamma_j^{-1}z}{z - \overline{\gamma}_j^{-1}}$$

belongs to H^∞. This implies the corollary.

4. Arguments of symbols

If T_ϕ is hyponormal, $|\phi| > 0$ a.e., and $\phi_0 = \phi/|\phi|$ then is T_{ϕ_0} hyponormal? In this section we answer this question. Statement (3) of Proposition 6 is a generalization of (4) of Theorem 5.

Lemma 2. *Suppose* $\log|\phi|$ *is integrable,* $\phi_0 = \phi/|\phi|$, *and* $s = h/|h|$ *where* h *is an outer function in* H^∞ *with* $|\phi| = |h|$. T_ϕ *is hyponormal if and only if* $\phi_0(\phi_0 - sg) = k$, *where* g *and* k *are in* H^∞, *and* $\|k\|_\infty \le 1$.

Proof. By Lemma 1, T_ϕ is hyponormal if and only if $\phi - g_1 = k\overline{\phi}$ where g_1 and k are in H^∞, and $\|k\|_\infty \le 1$. Since $\phi = \phi_0 \overline{s} h$, $\phi - g_1 = k\overline{\phi}$ if and only if $\phi_0(\phi_0 - sg_1/h) = k$. Since g_1/h is in H^∞, this implies the lemma.

Proposition 6. *Suppose* $|\phi|$ *is positive almost everywhere and* $\phi_0 = \phi/|\phi|$.

(1) *If* T_ϕ *is hyponormal and* $\log|\phi|$ *is not integrable then* T_{ϕ_0} *is hyponormal.*

(2) *Suppose* $|\phi| = |h|$ *for some outer* h *in* H^∞ *and* $h/|h|$ *is inner. If* T_ϕ *is hyponormal then* T_{ϕ_0} *is hyponormal.*

(3) *Suppose* ϕ *and* $\overline{\phi}$ *are of bounded type and* $\phi_0^2 \notin H^\infty$. *Then if* T_ϕ *is hyponormal,* T_{ϕ_0} *is not hyponormal.*

Proof. (1) By (1) of Theorem 4, $\phi = q\overline{\phi}$ for some inner q and hence $\phi_0 = q\overline{\phi}_0$. This implies that T_{ϕ_0} is hyponormal.

(2) In Lemma 2, s is inner by the hypothesis on $h/|h|$ and hence T_{ϕ_0} is hyponormal.

(3) Under the conditions on ϕ, $\overline{\phi}$, and ϕ_0, if both T_ϕ and T_{ϕ_0} are hyponormal then we claim a contradiction. By Lemma 1, there exist four functions k_1, k_2, g_1, and g_2 in H^∞ such that $\phi = k_1\overline{\phi} + g_1$ and $\phi_0 = k_2\overline{\phi}_0 + g_2$. If g_2 is zero then this contradicts that ϕ_0^2 is not inner. If g_2 is nonzero then

$$\phi = k_2\overline{\phi} + g_2|\phi|.$$

Since both ϕ and $\overline{\phi}$ are of bounded type, $|\phi|$ is also of bounded type and hence ϕ_0 is of bounded type. Therefore $\phi_0 = \overline{q}b$ where q and b are inner. By (3) of Theorem 5 ϕ_0 is inner. This contradicts that ϕ_0^2 is not inner, too.

5. Bounded type symbols

Abrahamse studied a hyponormal Toeplitz operator T_ϕ when ϕ or $\overline{\phi}$ is of bounded type. He gave several sufficient conditions [1, Proposition 1]. In this section we give necessary conditions, partially solving Problem 1 in [1]. If ϕ is of bounded type then ϕ has the form $\phi = \overline{q_1}q_2h$ where q_1 and q_2 are relatively prime inner functions, and h is outer. We call q_0 a minimal inner function for ϕ when q_0 is inner and it satisfies the following: if q is inner with $q\phi \in H^\infty$ then $\overline{q_0}q \in H^\infty$. In fact q_1 is the minimal inner function for ϕ.

Proposition 7. *Suppose* ϕ *is of bounded type, that is,* ϕ *has the form* $\phi = \overline{q_1}q_2h$ *where* q_1 *and* q_2 *are relatively prime inner functions, and* h *is outer.*

(1) *If* T_ϕ *is hyponormal then* $h/\overline{h}$ *has the form* $h/\overline{h} = \overline{F}G$ *where* F *and* G *are inner, and* $G\overline{q}_1^2$ *is in* H^∞.

(2) *If* $h/\overline{h} = \overline{F}G$ *where* F *and* G *are inner, and both* $G\overline{q}_1^2$ *and* $\overline{F}q_2^2$ *are in* H^∞, *then* T_ϕ *is hyponormal.*

Proof. (1) By Lemma 1 and the hypothesis on ϕ, $\overline{q}_1 q_2 h - g = k q_1 \overline{q}_2 \overline{h}$ where g and k are in H^∞ and hence'

$$q_2^2 - q_1 q_2 g/h = k q_1^2 \overline{h}/h = k q_1^2 \overline{G} F$$

where G and F are inner. Therefore $G(q_2^2 - q_1 q_2 g/h) = k q_1^2 F$ and hence $G\overline{q}_1^2 \in H^\infty$ becasue q_1 and q_2 are relatively prime.

(2) If $h/\overline{h} = \overline{F}G$ then by the hypothesis on F and G,

$$\frac{\phi}{\overline{\phi}} = \frac{\overline{q}_1 q_2 h}{q_1 \overline{q}_2 \overline{h}} = (\overline{q}_1^2 G)(q_2^2 \overline{F}) \in H^\infty.$$

Lemma 1 implies that T_ϕ is hyponormal.

Corollary 5. *Let* m *and* n *be nonnegative integers. Suppose* $\phi = \sum_{j=-n}^{m} a_j z^j$, $a_{-n} \neq 0$, *and* $a_m \neq 0$; *then the following hold.*

(1) *If* T_ϕ *is hyponormal then* $m \geq n$.

(2) *Suppose* $m \geq n$ *and if* $z^n\phi$ *is zero on a point* z_0 *outside the closed unit disc, then it is zero on the inverse of* $\overline{z}_0$ *with higher multiplicity. Then* T_ϕ *is hyponormal.*

(3) *Suppose* $m = n$ *and* T_ϕ *is not normal. If* $z^n\phi$ *does not have any zeros in the open unit disc then* T_ϕ *is not hyponormal.*

(4) *Suppose* $m = n$ *and if* $z^n\phi$ *has zeros only on the unit circle, then* T_ϕ *is normal.*

Proof. Put $k = z^n\phi = qh$ where q is inner and h is outer, and write

$$k = \prod_{j=1}^{l}(z - \alpha_j)\prod_{j=1}^{t}(z - \beta_j)\prod_{j=1}^{s}(z - \gamma_j)$$

where $|\alpha_j| < 1$ for $1 \leq j \leq l$, $|\beta_j| = 1$ for $1 \leq j \leq t$, and $|\gamma_j| > 1$ for $1 \leq j \leq s$. Then

$$q = \prod_{j=1}^{l}\frac{z - \alpha_j}{1 - \overline{\alpha}_j z}$$

and

$$h = \prod_{j=1}^{l}(1 - \overline{\alpha}_j z)\prod_{j=1}^{t}(z - \beta_j)\prod_{j=1}^{s}(z - \gamma_j),$$

and the degree of $k = m + n = l + t + s$. Hence

$$\frac{h}{\overline{h}} = \prod_{j=1}^{t}(-\beta_j)\prod_{j=1}^{s}\frac{\gamma_j}{\overline{\gamma}_j} z^{m+n}\prod_{j=1}^{l}\frac{1 - \overline{\alpha}_j z}{z - \alpha_j}\prod_{j=1}^{s}\frac{1 - \gamma_j^{-1} z}{z - \overline{\gamma}_j^{-1}}.$$

(1) If T_ϕ is hyponormal then by (1) of Proposition 7 z^{m+n}/z^{2n} is analytic and hence $m \geq n$.

(2) By the hypothesis $\alpha'_j\overline{\gamma}_j = 1$ for $1 \leq j \leq s \leq l$ where $\{\alpha'_j\}_{j=1}^{l} = \{\alpha_j\}_{j=1}^{l}$, and hence

$$\left(\prod_{j=1}^{s}\frac{1 - \gamma_j^{-1} z}{z - \overline{\gamma}_j^{-1}}\right) q \in H^\infty.$$

Then (2) of Proposition 7 implies (2).

(3) If $m = n$ and $k = \prod_{j=1}^{t}(z-\beta_j)\prod_{j=1}^{s}(z-\gamma_j)$ then

$$\overline{\phi} = z^n \prod_{j=1}^{t}(\overline{z}-\overline{\beta}_j)\prod_{j=1}^{s}(\overline{z}-\overline{\gamma}_j) = \overline{z}^n k',$$

where

$$k' = \prod_{j=1}^{t}(1-\overline{\beta}_j z)\prod_{j=1}^{s}(1-\overline{\gamma}_j z).$$

By (2) T_{ϕ}^* is hyponormal and hence if T_ϕ is not normal then it is not hyponormal.

(4) is clear by (2) and (3)

When $\phi = a_{-1}\overline{z} + a_0 + a_1 z$ and $a_{-1} \neq 0$, it is easy to see that T_ϕ is hyponormal if and only if $|a_1| \geq |a_{-1}|$. When $\phi = \overline{z}(z-\alpha)(z-\gamma)$ and $\alpha\gamma \neq 0$, T_ϕ is hyponormal if and only if $|\alpha\gamma| \leq 1$. Therefore the question about polynomials is still open.

Corollary 6. *Suppose ϕ is a rational function in L^∞: $\phi = k_2/k_1$ where k_1 and k_2 are relatively prime analytic polynomials with the same degree. Write*

$$k_1 = \prod_{j=1}^{l}(z-a_j)\prod_{j=1}^{n}(z-c_j)$$

where $|a_j| < 1$ for $1 \leq j \leq l$ and $|c_j| > 1$ for $1 \leq j \leq n$, and

$$k_2 = \prod_{j=1}^{d}(z-\alpha_j)\prod_{j=1}^{e}(z-\beta_j)\prod_{j=1}^{f}(z-\gamma_j)$$

where $|\alpha_j| < 1$ for $1 \leq j \leq d$, $|\beta_j| = 1$ for $1 \leq j \leq e$, and $|\gamma_j| > 1$ for $1 \leq j \leq f$. If T_ϕ is hyponormal then $\{a_j^{-1}\}_{j=1}^{l} \subseteq \{\overline{c}_j\}_{j=1}^{n}$. If $\{a_j^{-1}\}_{j=1}^{l} \subseteq \{\overline{c}_j\}_{j=1}^{n}$ and $\{\alpha_j^{-1}\}_{j=1}^{d} \supset \{\overline{\gamma}_j\}_{j=1}^{f}$ then T_ϕ is hyponormal.

Proof. Suppose $k_j = q_j h_j$ $(j = 1, 2)$ are inner outer factorizations. Then

$$q_1 = \prod_{j=1}^{l}\frac{z-a_j}{1-\overline{a}_j z}, \qquad h_1 = \prod_{j=1}^{l}(1-\overline{a}_j z)\prod_{j=1}^{n}(z-c_j)$$

and

$$q_2 = \prod_{j=1}^{d}\frac{z-\alpha_j}{1-\overline{\alpha}_j z}, \quad h_2 = \prod_{j=1}^{d}(1-\overline{\alpha}_j z)\prod_{j=1}^{e}(z-\beta_j)\prod_{j=1}^{f}(z-\gamma_j),$$

$\phi = \overline{q}_1 q_2 h$ and $h = h_2/h_1$. Since

$$\frac{\overline{h}_1}{h_1} = A\overline{z}^{l+n} q_1 \prod_{j=1}^{n}\frac{z-\overline{c}_j^{-1}}{1-c_j^{-1}z}$$

and

$$\frac{h_2}{\overline{h}_2} = B z^{d+e+f}\overline{q}_2 \prod_{j=1}^{f}\frac{1-\gamma_j^{-1}z}{z-\overline{\gamma}_j^{-1}}$$

where A and B are constants, $l+n=d+e+f$, $h/\overline{h}=\overline{F}G$, and

$$\overline{F}=C\overline{q}_2\prod_{j=1}^{f}\frac{1-\gamma_j^{-1}z}{z-\overline{\gamma}_j^{-1}} \quad \text{and} \quad G=q_1\prod_{j=1}^{n}\frac{z-\overline{c}_j^{-1}}{1-c_j^{-1}z}$$

where C is a constant with absolute value 1. Now Proposition 7 implies the corollary.

6. Self-commutator of T_ϕ

For ϕ in L^∞, put

$$\mathscr{E}(\phi)=\{k\in H^\infty : \phi=k\overline{\phi}+g,\ \ g\in H^\infty,\ \text{ and } \|k\|_\infty\le 1\}.$$

If $\mathscr{E}(\phi)$ contains an inner function q then by Corollary 2 ϕ has a simple form. In this section we study when $\mathscr{E}(\phi)$ contains an inner function.

Proposition 8. *Suppose $\mathscr{E}(\phi)$ contains at least two elements. Then the following are valid.*

(1) *ϕ is of bounded type.*

(2) *For any fixed function k in $\mathscr{E}(\phi)$*

$$\mathscr{E}(\phi)=\{k+q_0f : \|k+q_0f\|_\infty\le 1 \text{ and } f\in H^\infty\}$$

where q_0 is the minimal inner function for $\overline{\phi}$.

(3) *There exists an inner function b in $\mathscr{E}(\phi)$.*

Proof. If k_1 and k_2 are in $\mathscr{E}(\phi)$ and $k_1\ne k_2$ then $\phi=k_1\overline{\phi}+g_1=k_2\overline{\phi}+g_2$ for some g_1 and g_2 in H^∞. Hence $(k_1-k_2)\overline{\phi}=g_2-g_1$ and this implies (1). Since $(k_1-k_2)\overline{\phi}\in H^\infty$, $q\overline{\phi}$ belongs to H^∞ where q is the inner part of k_1-k_2. If q_0 is the minimal inner function for $\overline{\phi}$ then $\overline{q}_0q\in H^\infty$ and hence $\overline{q}_0(k_1-k_2)\in H^\infty$. This implies (2). If $\mathscr{E}(\phi)$ contains at least two functions then by (2) and a well-known theorem of Adamyan, Arov, and Krein (cf. [8, Theorem 5.3]) there exists an inner function in $\mathscr{E}(\phi)$.

We use Hankel operators. For $\phi\in L^\infty$, the Hankel operator $H_\phi : H^2\to L^2\ominus H^2$ is defined by $H_\phi f=(I-P)(\phi f)$ for $f\in H^2$, where P is the orthogonal projection of L^2 onto H^2 (cf. [14]).

Proposition 9. *Suppose ϕ is not bounded type and T_ϕ is hyponormal. If*

$$\operatorname{Ker}[T_\phi^*, T_\phi]\ne\{0\}$$

then there exists an inner function in $\mathscr{E}(\phi)$.

Proof. Suppose $x\in\operatorname{Ker}[T_\phi^*, T_\phi]$ and $k\in\mathscr{E}(\phi)$. Then $\|H_{\overline{\phi}}x\|_2=\|H_\phi x\|_2$ because $T_\phi^*T_\phi-T_\phi T_\phi^*=H_{\overline{\phi}}^*H_{\overline{\phi}}-H_\phi^*H_\phi$ and

$$\|H_\phi x\|_2=\|H_{k\overline{\phi}}x\|_2\le\|kH_{\overline{\phi}}x\|_2\le\|H_{\overline{\phi}}x\|_2.$$

Since ϕ is not of bounded type, $\|H_\phi x\|\ne 0$ and hence $|k|=1$ a.e.

The following lemma is known and is easy to prove.

Lemma 3. *For* $k \in \mathscr{E}(\phi)$,

$$[T_\phi^*, T_\phi] = H_{\overline{\phi}}^*(1 - JT_{\tilde{k}}T_{\tilde{k}}^*J)H_{\overline{\phi}}$$

where $\tilde{k}(z) = \overline{k(\overline{z})}$ *and* $(Jx)(z) = x(\overline{z})\overline{z}$ *for* x *in* L^2.

Theorem 10. T_ϕ *is hyponormal and* $[T_\phi^*, T_\phi]$ *is a finite rank operator if and only if there exists a finite Blaschke product* b *in* $\mathscr{E}(\phi)$. *Then we can choose* b *such that the degree of* $b = \operatorname{rank}[T_\phi^*, T_\phi]$.

Proof. If $b \in \mathscr{E}(\phi)$ is a finite Blaschke product of degree n then

$$\operatorname{rank}(1 - JT_{\tilde{b}}T_{\tilde{b}}^*J|\overline{z}\overline{H}^2) \leq n$$

and hence by Lemma 3 $\operatorname{rank}[T_\phi^*, T_\phi] \leq n$. We will show the existence of a finite Blaschke product b in $\mathscr{E}(\phi)$ of degree n, assuming that $k \in \mathscr{E}(\phi)$ and $\operatorname{rank}[T_\phi^*, T_\phi] = n < \infty$. If $\overline{\phi}$ is not of bounded type then $\operatorname{Ran} H_{\overline{\phi}}$ is dense in $\overline{z}\overline{H}^2$ and hence by Lemma 3 $\operatorname{rank}[T_\phi^*, T_\phi] = \operatorname{rank}(1 - JT_{\tilde{k}}T_{\tilde{k}}^*J|\overline{z}\overline{H}^2)$. Hence $\tilde{k}$ is a finite Blaschke product of degree n and so is k. Now we assume that $\overline{\phi}$ is of bounded type. Then by Beurling's theorem $\operatorname{Ker} H_{\overline{\phi}} = qH^2$ for some inner q. By Lemma 3 $\operatorname{Ker}[T_\phi^*, T_\phi] \supseteq \operatorname{Ker} H_{\overline{\phi}}$.

Case (i). $\operatorname{Ker}[T_\phi^*, T_\phi] = \operatorname{Ker} H_{\overline{\phi}}$. Then the closure of $\operatorname{Ran}[T_\phi^*, T_\phi] = H^2 \ominus qH^2$ and hence

$$\dim(H^2 \ominus qH^2) = \operatorname{rank}[T_\phi^*, T_\phi] = n < \infty.$$

Therefore q is a finite Blaschke product of degree n. By Pick's theorem (cf. [8, Theorem 2.2]), there exists a finite Blaschke product b such that $b \in k + qH^\infty$ and the degree of b is at most n. $\operatorname{Ker} H_{\overline{\phi}} = qH^2$ implies that q is the minimal inner function for $\overline{\phi}$, and hence by (2) of Proposition 8 b belongs to $\mathscr{E}(\phi)$.

Case (ii). $\operatorname{Ker}[T_\phi^*, T_\phi] \supsetneq \operatorname{Ker} H_{\overline{\phi}}$. Then there exists a function x in $\operatorname{Ker}[T_\phi^*, T_\phi]$ such that $\|H_{\overline{\phi}}x\| \neq 0$. The proof of Proposition 9 implies that k is inner. Since $\operatorname{Ker} H_{\overline{\phi}}^* = \overline{q}\overline{z}\overline{H}^2 =$ the orthogonal complement of $\operatorname{Ran} H_{\overline{\phi}}$ in $\overline{z}\overline{H}^2$, by Lemma 3

$$\begin{aligned}\operatorname{rank}[T_\phi^*, T_\phi] &= \dim(1 - JT_{\tilde{k}}T_{\tilde{k}}^*J)H_{\overline{\phi}}H^2 \\ &= \dim(1 - T_{\tilde{k}}T_{\tilde{k}}^*)(H^2 \ominus \tilde{q}H^2) \\ &= \dim\{H^2 \ominus \tilde{q}H^2/(H^2 \ominus \tilde{q}H^2) \cap \tilde{k}H^2\}.\end{aligned}$$

By a theorem of the first author [11], k is a finite Blaschke product of degree n.

Corollary 7. *If* ϕ *is a trigonometric polynomial and* T_ϕ *is hyponormal, then there exists a finite Blaschke product in* $\mathscr{E}(\phi)$.

7. Kernel of the self-commutator of T_ϕ

In this section we are interested in describing $\operatorname{Ker}[T_\phi^*, T_\phi]$ when T_ϕ is hyponormal. $\mathscr{E}(\phi) \ni 0$ if and only if $\phi \in H^\infty$. Then $\operatorname{Ker}[T_\phi^*, T_\phi] = \{0\}$ or

$\operatorname{Ker}[T_\phi^*, T_\phi] = qH^2$ for some inner function q. Proposition 11 is a generalization of this. There is a constant β in $\mathscr{E}(\phi)$ with $|\beta| = 1$ if and only if $\phi = \beta^{1/2}u + \alpha$ where β and α are constants and $|\beta| = 1$. Then $\operatorname{Ker}[T_\phi^*, T_\phi] = H^2$. Proposition 12 is a generalization of this.

Proposition 11. *If there exists a function in $\mathscr{E}(\phi)$ which is not inner, then $\operatorname{Ker}[T_\phi^*, T_\phi] = qH^2$ for some inner q or $\operatorname{Ker}[T_\phi^*, T_\phi] = \{0\}$. Hence if $\mathscr{E}(\phi)$ contains at least two functions then $\operatorname{Ker}[T_\phi^*, T_\phi] = qH^2$.*

Proof. Let k be not an inner function in $\mathscr{E}(\phi)$. Then $\operatorname{Ker}(1 - T_{\tilde{k}}T_{\tilde{k}}^*) = \{0\}$ because $\tilde{k}$ is not inner. Hence by Lemma 3 $\operatorname{Ker}[T_\phi^*, T_\phi] = \operatorname{Ker} H_{\overline{\phi}}$ and this implies the proposition.

Proposition 12. *If there exists an inner function q in $\mathscr{E}(\phi)$ then*

$$\operatorname{Ker}[T_\phi^*, T_\phi] = \{x \in H^2 \colon T_{\overline{\phi}}(qx) \in qH^2\} = \operatorname{Ker}(T_{\overline{\phi}}T_q - T_q T_{\overline{\phi}}).$$

Proof. It is sufficient to prove that $\operatorname{Ran}[T_\phi^*, T_\phi]$ is dense in $T_{\phi\overline{q}}(H^2 \ominus qH^2)$. Since

$$\begin{aligned} H_{\overline{\phi}}^* J(1 - T_{\tilde{q}}T_{\tilde{q}}^*)H^2 &= H_{\overline{\phi}}^* J(H^2 \ominus \tilde{q}H^2) = P\phi(\overline{zH}^2 \ominus \overline{qzH}^2) \\ &= P\phi\overline{q}(H^2 \ominus qH^2) = T_{\phi\overline{q}}(H^2 \ominus qH^2), \end{aligned}$$

by Lemma 3 $\operatorname{Ran}[T_\phi^*, T_\phi]$ is dense in $T_{\phi\overline{q}}(H^2 \ominus qH^2)$. The second equality is trivial.

Lemma 4. *$H_\phi^* H_\phi = H_\psi^* H_\psi$ if and only if $\phi - c\psi \in H^\infty$ for some constant c with $|c| = 1$.*

Proof. If $H_\psi^* H_\psi - H_\phi^* H_\phi \geq 0$ then by the proof of [5, Theorem 1] there exist two functions k and g in H^∞ with $\|k\|_\infty \leq 1$ such that $\phi = k\psi + g$. Lemma 1 is the special case: $\psi = \overline{\phi}$. Hence $H_\phi^* H_\phi = H_\psi^* H_\psi$ then there exist two other functions h and f in H^∞ with $\|h\|_\infty \leq 1$ such that $\psi = h\phi + f$. Then $(1 - kh)\phi = kf + g$. If $1 - kh$ is nonzero then $1 - kh$ is outer and hence ϕ belongs to H^∞. Similarly $\psi \in H^\infty$. If $1 = kh$ then both k and h are constants with $|k| = |h| = 1$. Hence $\phi - c\psi \in H^\infty$ for some constant c with $|c| = 1$. The converse is clear.

Theorem 13. *Let T_ϕ be hyponormal. Then $\operatorname{Ker}[T_\phi^*, T_\phi] \supseteq qH^2$ for some inner function q if and only if $\overline{\phi}$ is of bounded type or T_ϕ is normal. Moreover if $\operatorname{Ker}[T_\phi^*, T_\phi] \supseteq qH^2$, and T_ϕ is not normal then $\operatorname{Ker} H_{\overline{\phi}} \supseteq qH^2$.*

Proof. By Lemma 3 $\operatorname{Ker}[T_\phi^*, T_\phi] \supseteq \operatorname{Ker} H_{\overline{\phi}}$ and hence the 'if' part is clear. We will show the 'only if' part. If $\operatorname{Ker}[T_\phi^*, T_\phi] \supseteq qH^2$ then

$$0 = T_q^*[T_\phi^*, T_\phi]T_q = T_{\phi q}^* T_{\phi q} - T_{\overline{\phi}q}^* T_{\overline{\phi}q} = H_{\overline{\phi}q}^* H_{\overline{\phi}q} - H_{\phi q}^* H_{\phi q}.$$

By Lemma 4 $\overline{\phi}q - c\phi q \in H^\infty$ for some constant c with $|c| = 1$. Since T_ϕ is hyponormal, $\phi - k\overline{\phi} = g \in H^\infty$ for some $k \in H^\infty$ with $\|k\|_\infty \leq 1$. Hence $\overline{\phi}(1 - ck)q \in H^\infty$. If $\overline{\phi}$ is not of bounded type then $1 - ck = 0$ and hence $k = \overline{c}$. By the remark above Lemma 4, T_ϕ is normal. If T_ϕ is not normal then $1 - ck$ is nonzero and hence $1 - ck$ is outer. Therefore $\overline{\phi}q \in H^2$ and hence $\operatorname{Ker} H_{\overline{\phi}} \supseteq qH^2$.

8. SUBNORMAL TOEPLITZ OPERATOR

In this section we will give a simple proof of the Abrahamse's theorem [1] which gives a sufficient condition for the positive answer of Halmos's question [9]. Then we will consider subnormal Toeplitz operators with self-commutator of finite rank.

Theorem 14 [Abrahamse]. *If T_ϕ is subnormal and if ϕ or $\overline{\phi}$ is of bounded type then T_ϕ is normal or analytic.*

Proof. If T_ϕ is subnormal then by Lemma 1 $\phi = k\overline{\phi} + g$, $k \in H^\infty$, and $g \in H^\infty$. If $\phi \notin H^\infty$ and ϕ is of bounded type then k is nonzero and hence $\overline{\phi}$ is also of bounded type (see [1, Lemma 6]). Therefore, we may assume that $\overline{\phi}$ is of bounded type. If T_ϕ is nonnormal then $qH^2 = \operatorname{Ker} H_{\overline{\phi}} \subseteq \operatorname{Ker}[T_\phi^*, T_\phi]$ and q is a nonconstant inner function. $T_\phi \operatorname{Ker}[T_\phi^*, T_\phi] \subseteq \operatorname{Ker}[T_\phi^*, T_\phi]$ because T_ϕ is subnormal [1], and hence $T_\phi(qH^2) \subseteq \operatorname{Ker}[T_\phi^*, T_\phi]$. While $\phi q \in H^\infty$ because $\operatorname{Ker} H_\phi \supseteq \operatorname{Ker} H_{\overline{\phi}}$, and hence the closure of $\phi q H^2$ is an invariant subspace in $\operatorname{Ker}[T_\phi^*, T_\phi]$. By Beurling's theorem and Theorem 13, $\phi q H^2 \subseteq \operatorname{Ker} H_{\overline{\phi}} = qH^2$ and hence $\phi \in H^\infty$.

Recall that a Toeplitz operator T_ϕ is hyponormal and $\operatorname{Ran}[T_\phi^*, T_\phi] < \infty$ if and only if $\phi = q\overline{\phi} + g$ where q is a finite Blaschke product and $g \in H^\infty$. The proof of Proposition 12 shows the following lemma.

Lemma 5. *If $\phi = q\overline{\phi} + g$ where q is inner and $g \in H^\infty$ then the closure of* $\operatorname{Ran}[T_\phi^*, T_\phi]$ *equals the closure of $T_{\phi\overline{q}}(H^2 \ominus qH^2)$.*

Lemma 6. *Suppose $\phi = q\overline{\phi} + g$ where q is inner and $g \in H^\infty$, and T_ϕ is subnormal.*

(1) *Put M = the closure of* $\operatorname{Ran}[T_\phi^*, T_\phi] + (H^2 \ominus qH^2)$ *then M is a $T_{\overline{\phi}}$-invariant subspace.*

(2) *If $\overline{\phi}$ is not of bounded type then* $\operatorname{Ran}[T_\phi^*, T_\phi] \cap (H^2 \ominus qH^2) = \{0\}$.

Proof. (1) Since $\phi = q\overline{\phi} + g$, $T_{\overline{\phi}}(H^2 \ominus qH^2) \subseteq T_{\overline{q}\phi}(H^2 \ominus qH^2) + T_{\overline{g}}(H^2 \ominus qH^2)$. Hence by Lemma 5

$$T_{\overline{\phi}}(H^2 \ominus qH^2) \subseteq \text{ the closure of } \operatorname{Ran}[T_\phi^*, T_\phi] + (H^2 \ominus qH^2).$$

This implies (1) because the closure of $\operatorname{Ran}[T_\phi^*, T_\phi]$ is a $T_{\overline{\phi}}$-invariant subspace.

(2) Since $\operatorname{Ran}[T_\phi^*, T_\phi] \subseteq T_{\phi\overline{q}}(H^2 \ominus qH^2)$ by the proof of Lemma 5, it is sufficient to prove that

$$T_{\phi\overline{q}}(H^2 \ominus qH^2) \cap (H^2 \ominus qH^2) = \{0\}.$$

If $x \in T_{\phi\overline{q}}(H^2 \ominus qH^2) \cap (H^2 \ominus qH^2)$ then $x = T_{\phi\overline{q}}y$ for some $y \in H^2 \ominus qH^2$. Since $T_{\overline{q}}x = 0$ because $x \in H^2 \ominus qH^2$,

$$T_{\phi\overline{q}}2y = T_{\overline{q}}T_{\phi\overline{q}}y = T_{\overline{q}}x = 0,$$

and hence $\phi\overline{q}^2 y = \overline{zk}$ for some $k \in H^2$. Therefore $\overline{\phi}q(q\overline{y}) = zk \in H^2$ and $q\overline{y} \in H^2$. If $\overline{\phi}$ is not of bounded type then $y = 0$ and hence $x = 0$. This implies (2).

Theorem 15. *If* T_ϕ *is subnormal and* $\phi = q\overline{\phi}$ *where* q *is a finite Blaschke product, then* T_ϕ *is normal or analytic.*

Proof. By Abrahamse's theorem we may assume that $\overline{\phi}$ is not of bounded type. Under this assumption, we show that q is constant and so T_ϕ is normal. Since $\phi = q\overline{\phi}$ and q is a finite Blaschke product, by Lemma 5

$$\mathrm{Ran}[T_\phi^*, T_\phi] = T_{\phi\overline{q}}(H^2 \ominus qH^2) = T_{\overline{\phi}}(H^2 \ominus qH^2).$$

Since T_ϕ is subnormal, $T_{\overline{\phi}}\mathrm{Ran}[T_\phi^*, T_\phi] \subseteq \mathrm{Ran}[T_\phi^*, T_\phi]$ and so

$$T_{\overline{\phi}}\{\mathrm{Ran}[T_\phi^*, T_\phi] + (H^2 \ominus qH^2)\} = \mathrm{Ran}[T_\phi^*, T_\phi].$$

Then by (2) of Lemma 6

$$\dim \mathrm{Ker}\, T_{\overline{\phi}}|\{\mathrm{Ran}[T_\phi^*, T_\phi] + (H^2 \ominus qH^2)\} = \dim(H^2 \ominus qH^2),$$

and hence $\dim \mathrm{Ker}\, T_{\overline{\phi}} \geq \dim(H^2 \ominus qH^2)$. The relation $\phi = q\overline{\phi}$ also implies

$$T_\phi(\mathrm{Ker}\, T_{\overline{\phi}}) \subseteq \mathrm{Ker}\, T_{\overline{q}} = H^2 \ominus qH^2.$$

Since T_ϕ is hyponormal, Ker $T_{\overline{\phi}} \supseteq \mathrm{Ker}\, T_\phi$ and $\mathrm{Ker}\, T_\phi = \{0\}$ by Coburn's theorem [7, Proposition 7.24]. Therefore we have $T_\phi(\mathrm{Ker}\, T_{\overline{\phi}}) = H^2 \ominus qH^2$ and so $\mathrm{Ran}\, T_\phi \supseteq H^2 \ominus qH^2$. Then, noting the relation $T_\phi T_q - T_q T_\phi = H_{\overline{q}}^* H_\phi$ and $\mathrm{Ran}\, H_{\overline{q}}^* = H^2 \ominus qH^2$, we see that the inclusion $\mathrm{Ran}\, T_\phi \supseteq q^n(H^2 \ominus qH^2)$ implies $\mathrm{Ran}\, T_\phi \supseteq q^{n+1}(H^2 \ominus qH^2)$ for $n = 0, 1, 2, \dots$. Thus we have $\mathrm{Ran}\, T_\phi \supseteq \bigcup_{n \geq 0} q^n(H^2 \ominus qH^2)$. Suppose q is not constant. Then $\bigvee_{n \geq 0} q^n(H^2 \ominus qH^2) = H^2$ and T_ϕ has dense range. On the other hand, the relation $T_\phi(\mathrm{Ker}\, T_{\overline{\phi}}) = H^2 \ominus qH^2$ implies $\mathrm{Ker}\, T_{\overline{\phi}} \neq \{0\}$. This is a contradiction. We conclude that q is constant.

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